Universidad Simón Bolívar
Departamento de Matemáticas
Puras y Aplicadas
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Nombre: ____________________

Carnet: ______________ Sección: _____

MA-3111 —Segundo parcial (35 %) —

1. *a*) Desarrolle en la serie de Fourier seno la función $f(x) = x\,,\ x \in [-\pi\,,\,0]$.
   Hacemos la extensión impar g de f al intervalo $[0,\pi] \Rightarrow g(x) = x,\ \ x \in [-\pi,\pi]$.
   $$g(x) \approx \sum_{h=1}^{+\infty} b_n \operatorname{sen} nx\,,\ \ b_n = \tfrac{2}{\pi}\int_0^{\pi} t \operatorname{sen} nt\, dt,\ \ x \in [-\pi,\pi].$$
   $$\Rightarrow b_n = \frac{2t}{\pi}\left(-\frac{\cos nt}{n}\right)|_0^{\pi} + \frac{2}{\pi n}\int_0^{\pi} \cos nt\, dt = \frac{2}{n}\cos n\pi = 2\frac{(-1)^{n+1}}{n}.$$
   $$\Rightarrow f(x) \approx \sum_{n=1}^{+\infty} \frac{2(-1)^{n+1}}{n} \operatorname{sen} nx\,,\ \ x \in [-\pi,0].$$

   b) a partir de este desarrollo calcule la suma $\displaystyle\sum_{n=1}^{+\infty}\frac{1}{n^2}$.
   $$g(x) \approx \sum_{n=1}^{+\infty} \frac{2(-1)^{n+1}}{n} \operatorname{sen} nx\,,\ \ x \in [-\pi,\pi]$$
   Por la igualdad de Parseval-Steklov
   $$\frac{1}{\pi}\int_{-\pi}^{\pi} |g(x)|^2 = \sum_{n=1}^{+\infty} |a_n|^2 = \sum_{n=1}^{+\infty} \frac{4}{n^2} \Rightarrow \sum_{1}^{+\infty} \frac{1}{n^2} = \frac{1}{4\pi}\int_{-\pi}^{\pi} x^2 dx = \frac{2}{4\pi}\int_0^{\pi} x^2 dx =$$
   $$= \frac{1}{2\pi}\cdot\frac{\pi^3}{3} = \frac{\pi^2}{6}.\ \text{Así}\ \sum_{n=1}^{+\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

   c) Diga a que converge la serie encontrada en el intervalo $[-\pi,+\pi]$. $x \in (-\pi,\pi) \Rightarrow \displaystyle\sum_{n=1}^{+\infty} a_n \operatorname{sen} nx = g(x)$, ya que g es diferenciable. Sea $x = \pm\pi \Rightarrow$ hacemos la extensión $2\pi-$ periódica $\tilde{g}$ de g.
   $$y = \tilde{g}(x) \Rightarrow S_n(-\pi) \underset{n\to\infty}{\longrightarrow} \frac{\tilde{g}(-\pi-0)+\tilde{g}(-\pi+0)}{2} = \frac{\pi+(-\pi)}{2} = 0\,,$$
   $$S_n(+\pi) \underset{n\to\infty}{\longrightarrow} \frac{\tilde{g}(\pi-0)+\tilde{g}(\pi+0)}{2} = \frac{\pi+(-\pi)}{2} = 0.$$
   $$\Rightarrow \sum_{n=1}^{+\infty} \frac{(-1)^{n+1}\,2}{n} \operatorname{sen} nx = \begin{cases} 0, & \text{si } x = \pm\pi \\ x, & \text{si } x \in (-\pi,\pi). \end{cases}$$

2. Hallar la solución del siguiente problema:

$$\begin{cases} u_{tt} &= 4u_{xx} \\ u(x,0) &= 0 \\ u_t(x,0) &= 8\pi \operatorname{sen} 4\pi x \\ u(0,t) &= u(3,t) = 0 \end{cases}$$, donde $u = u(x,t)$, $t \geq 0, \;\; x \in [0,3]$.

$$\begin{aligned} u(x,t) &= X(x)T(t) \\ u(0,t) &= 0 \Rightarrow X(0)T(t) = 0 \Rightarrow X(0) = 0. \\ u(3,t) &= 0 \Rightarrow X(3)T(t) = 0 \Rightarrow X(3) = 0. \\ u_{tt} &= 4u_{xx} \Rightarrow T''(t)X(x) = 4X''(x)T(t) \Rightarrow \frac{T''(t)}{4T(t)} = \frac{X''(x)}{X(x)} =: -\lambda. \\ &\Rightarrow T''(t) + 4\lambda T(t) = 0\,, \;\; X''(x) + \lambda X(x) = 0. \\ &\Rightarrow \begin{cases} X''(x) + \lambda X(x) = 0 \\ X(0) = X(3) = 0 \end{cases} \end{aligned}$$

1) $\lambda < 0 \Rightarrow X(x) = c_1 e^{\sqrt{-\lambda}x} + c_2 e^{-\sqrt{-\lambda}x}$
$X(0) = 0 \Rightarrow c_1 + c_2 = 0 \Rightarrow c_2 = -c_1;$
$X(3) = 0 \Rightarrow c_1 \left(e^{\sqrt{-\lambda}3} - e - \sqrt{-\lambda}3\right) = c_1\, e^{\sqrt{-\lambda}3}\left(e^{2-\sqrt{-\lambda}3} - 1\right) = 0 \;\Rightarrow c_1 = 0, \Rightarrow$
$c_2 = 0 \Rightarrow X \equiv 0.$

2) $\lambda = 0 \Rightarrow X''(x) = 0 \Rightarrow X(x) = c_1 x + c_2$
$X(0) = 0 \Rightarrow c_2 = 0\,,\; X(3) = 0 \Rightarrow 3c_1 = 0 \Rightarrow c_1 = 0 \Rightarrow X \equiv 0.$

3) $\lambda > 0 \Rightarrow X(x) = c_1 \cos\sqrt{\lambda}\,x + c_2 \operatorname{sen}\sqrt{\lambda}\,x$
$X(0) = 0 \Rightarrow c_1 = 0\,,\;\; X(3) = 0 \Rightarrow c_2 \operatorname{sen}\sqrt{\lambda}3 = 0 \Rightarrow 3\sqrt{\lambda} = \pi n,\;\; n = 1, 2, \ldots$
$\Rightarrow \lambda = \lambda_n = (\pi n/3)^2\,,\;\; X = X_n(x) = c_n \operatorname{sen}\sqrt{\lambda_n}\,x = c_n \operatorname{sen}\dfrac{\pi n}{3}\,x.$
$T''(t) + 4\lambda_n T(t) = 0 \Rightarrow T_n(t) = A_n \cos\sqrt{4\lambda_n}\,t + B_n \operatorname{sen}\sqrt{4\lambda_n}\,t.$
$\Rightarrow u = u_n(x,t) = T_n(t)X_n(x) = (A_n \cos\sqrt{4\lambda_n}\,t + B_n \operatorname{sen}\sqrt{4\lambda_n}\,t)c_n \operatorname{sen}\sqrt{\lambda_n}\,x$
$\Rightarrow u_n(x,t) = (a_n \cos 2\sqrt{\lambda_n}\,t + b_n \operatorname{sen} 2\sqrt{\lambda_n}\,t) \operatorname{sen}\sqrt{\lambda_n}\,x.$

Sea $u(x,t) = \displaystyle\sum_{n=1}^{+\infty} u_n(x,t) = \sum_{n=1}^{+\infty} (a_n \cos 2\sqrt{\lambda_n}\,t + b_n \operatorname{sen} 2\sqrt{\lambda_n}\,t) \operatorname{sen}\sqrt{\lambda_n} x$

$u(x,0) = 0 \Rightarrow \displaystyle\sum_{n=1}^{+\infty} a_n \operatorname{sen}\sqrt{\lambda_n}\,x = 0 \Rightarrow a_n = 0.$

$u_t(x,0) = 8\pi \operatorname{sen} 4\pi x \Rightarrow \displaystyle\sum_{n=1}^{+\infty} b_n 2\sqrt{\lambda_n}\, \operatorname{sen}\sqrt{\lambda_n}\,x = 8\pi \operatorname{sen} 4\pi x.$

$\Rightarrow \displaystyle\sum_{n=1}^{+\infty} b_n \cdot 2 \cdot \frac{\pi n}{3} \cdot \operatorname{sen}\frac{\pi n}{3}\,x = 8\pi \operatorname{sen} 4\pi x.$

$\dfrac{\pi n}{3}\,x = 4\pi x$ si $n = 12 \Rightarrow b_n = 0\,,\;\; n \neq 12.$
$n = 12 \Rightarrow b_{12} \cdot 8\pi \operatorname{sen} 4\pi x = 8\pi \operatorname{sen} 4\pi x \Rightarrow b_{12} = 1.$

$$\Rightarrow u(x,t) = b_n \operatorname{sen} 2\sqrt{\lambda_n}\, t \operatorname{sen} \sqrt{\lambda_n}\, x \Big|_{n=12} = \operatorname{sen} \frac{2\pi n}{3}\, t \operatorname{sen} \frac{\pi n}{3} x \Big|_{n=12}$$

$$\boxed{u(x,t) = \operatorname{sen} 8\pi t \cdot \operatorname{sen} 4\pi x.}$$

3. Halle la transformada de Fourier de la función $f(t) = (4-3t)e^{-(t+2)^2}$
   Solución:

$$\begin{aligned}
\hat{f} &= \mathcal{F}\left(4e^{-(t+2)^2} - 3te^{-(t+2)}\right)[\omega] \\
&= 4 \cdot \mathcal{F}\left(e^{-(t+2)^2}\right)[\omega] - 3\mathcal{F}\left(te^{-(t+2)^2}\right)[\omega] \\
&= 4\mathcal{F}\left(e^{-(t-(-2))^2}\right)[\omega] - 3\frac{id}{d\omega}\mathcal{F}\left(e^{-(t-(-2))^2}\right)[\omega] \\
&= 4e^{-i\omega(-2)}(e^{-t^2})[\omega] - 3\frac{id}{d\omega}\left(e^{-i\omega(-2)}\mathcal{F}(e^{-t^2})[\omega]\right) , \\
& \quad \mathcal{F}(e^{-t^2})[\omega] = \mathcal{F}\left(e^{-2t^2/2}\right)[\omega] = \frac{1}{\sqrt{2\pi \cdot 2}}\, e^{-\frac{\omega^2}{2\cdot 2}} = \frac{1}{\sqrt{4\pi}}\, e^{-\omega^2/4} \\
\Rightarrow & \quad \hat{f}(\omega) = 4e^{2i\omega} \cdot \frac{1}{\sqrt{4\pi}}\, e^{-\omega^2/4} - 3\frac{id}{d\omega}\left(e^{2i\omega} \cdot \frac{1}{\sqrt{4\pi}}\, e^{-\omega^2/4}\right) \\
&= \frac{2e^{2i\omega}}{\sqrt{\pi}}\, e^{-\omega^2/4} - \frac{3i}{\sqrt{4\pi}}\left(2ie^{2i\omega} \cdot e^{-\omega^2/4} - \frac{\omega}{2}e^{2i\omega}\, e^{\omega^2/4}\right) \\
&= \left(2 + 3 + \frac{3i\omega}{4}\right)\frac{e^{2i\omega}e^{-\omega^2/4}}{\sqrt{\pi}} = \left(5 + \frac{3}{4}i\omega\right)\frac{e^{2i\omega}e^{-\omega^2/4}}{\sqrt{\pi}}.
\end{aligned}$$

4. Halle explícitamente la solución $u = u(x,t)$ del siguiente problema

$$\begin{cases} u_t &= 5u_{xx} \\ u(x,0) &= 3\delta(x) \end{cases} , \text{ donde } \quad x \in \mathbb{R}, \quad t \geq 0.$$

Solución:] $\hat{u}(\omega,t)=\mathcal{F}_{x\to\omega}(u(x,t))$.

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\begin{aligned}
u_t &= 5u_{xx} \Rightarrow (\mathcal{F}_{x\to\omega}(u(x,t)))_t = 5\mathcal{F}_{x\to\omega}(u_{xx}(x,t))\\
&\Rightarrow \hat{u}_t(\omega,t) = 5(i\omega)^2\hat{u}(\omega,t) \Rightarrow \hat{u}_t + 5\omega^2\hat{u} = 0.\\
& \quad u(x,0) = 3\delta(x) \Rightarrow \mathcal{F}_{x\to\omega}(u(x,0)) = 3\mathcal{F}_{x\to\omega}(\delta(x))\\
&\Rightarrow \hat{u}(\omega,0) = 3\cdot\frac{1}{2\pi} \Rightarrow \hat{u}(\omega,0) = \frac{3}{2\pi}.\\
&\Rightarrow \begin{cases} \hat{u}_t + 5\omega^2\hat{u} = 0 \quad , \quad \hat{u} = \hat{u}(\omega,t). \\ \hat{u}(\omega,t)|_{t=0} = \frac{3}{2\pi} \end{cases}\\
&\Rightarrow \hat{u}(\omega,t) = \frac{3}{2\pi}e^{-5\omega^2 t}.\\
&\Rightarrow u(x,t) = \mathcal{F}^{-1}_{\omega\to x}(\hat{u}(\omega,t)) = \frac{3}{2\pi}\int_{-\infty}^{+\infty} e^{-5\omega^2 t}e^{i\omega x}d\omega\\
&= 3\cdot\frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{-5\omega^2 t}\,e^{-i\omega(-x)}d\omega = 3\mathcal{F}_{\omega\to x}(e^{-5\omega^2 t})[-x]\\
&= 3\mathcal{F}_{\omega\to x}(e^{-10t\omega^2/2})[-x] = 3\cdot\frac{1}{\sqrt{2\pi\cdot 10t}}\,e^{-x^2/(2\cdot 10t)}|_{-x}\\
&\Rightarrow \boxed{u(x,t) = \frac{3}{\sqrt{20\pi t}}\,e^{-\frac{x^2}{20t}}}.
\end{aligned}
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